28.02.2011 **Solution**

Advanced Functional Analysis - Mid-Sem Exam - Semester II

1. **Question:** Let X be a topological vector space. $A \subset X$ is said to be totally bounded, if for any neighborhood $\cup$ of 0, there exists a finite set $F \subset A$ such that $A \subset (F + \cup)$. Show that any totally bounded set is bounded and any compact set is bounded.

   Solution: Let A be a totally bounded set in a topological vector space X and U be an arbitrary open set containing 0. Choose a balanced nbhd W of 0 such that $W \subset U$. Since A is totally bounded there exists a finite set $F = \{x_1, x_2, ..., x_n\}$ such that $A \subset F + W$. Choose a large $s > 0$ such that $x_i \in (s-1)W \quad \forall i = 1, 2, ..., n$.
   i.e $F \subset (s-1)W$, which implies that $A \subset sW$. For $t > s$, we have $A \subset tW \subset tU$. Hence A is bounded. □

2. **Question:**

   Consider $C[0,1]$. Show that there is a locally convex topological vector space topology on $C[0,1]$ that is different from the norm topology. Give all the details of your answer.

   Solution: Define $\|f\|_2 = \int_{[0,1]} |f(x)|^2 dx$, then this defines a norm on $C[0,1]$. $C[0,1]$ is not complete with respect to this norm.

   □

3. **Question:**

   Consider $l^2 = \big\{\{\alpha_n\}_{n\geq 1} : \Sigma \mid \alpha_n \mid^2 \quad < \infty\big)\big\}$. Let $U \subset l^2$ be a proper neighborhood of 0. Show that there exists a sequence $x = \{\alpha_n\}_{n\leq 1} \; \varepsilon \; U$ with infinitely many non-zero coordinates.

   Solution: Let $U \subset l^2$ be a proper nbhd of 0. Then there exists a $\varepsilon > 0$ such that $B(0,\varepsilon) = \{x \in l^2 : \|x\| < \varepsilon\} \subset U$. Consider $y = (\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2^2}}, ..., \sqrt{\frac{1}{2^n}}, ...)$, $\|y\|^2 = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ i.e $y \in l^2$ with $\|y\| = 1$. It follows that $\frac{\varepsilon}{2} y \in B(0,\varepsilon) \subset U$ and $\frac{\varepsilon}{2} y$ has infinitely many non-zero coordinates. □

4. **Question:**

   Let $c_0 = \{\{\alpha_n\}_{n\leq 1} : \lim \alpha_n = 0\}$. Let d be an invariant matric on c_0 making it a topological vector space. Let d' denote the supremum metric on c_0. Suppose that topologies generated by d and d' are the same Show that d is a complete metric.

   Solution: Let $\mathbb{B}$ and $\mathbb{B}'$ be the local bases for the topologies generated by the metrics d and d' respectively. We will show that every d-cauchy sequence is also a d'-cauchy sequence. Let x_n be a d-cauchy sequence. Let $V' \in \mathbb{B}'$, then there exists a $U \in \mathbb{B}$ such that $U \subset V'$. Since x_n d-cauchy sequence there exists a N such that $x_n - x_m \in U$ for all $n, m > N$. It follows that x_n is d'-cauchy. This implies that d is a complete metric since d' is.

   □

5. **Question:**

   Let $l^1 = \{\{\alpha_n\}_{n\leq 1} : \Sigma \mid \alpha_n \mid < \infty\}$ with the usual norm. Show that this space is separable when equipped with the weak topology.

   Solution: $x_n \to^w x \Leftrightarrow x_n \to x$ in l^1(Schur's lemma). Since $(l^1, \|\|_1)$ is separable and dense subsets of norm and weak topology are the same, it follows that l^1 is separable in weak topology. □

6. **Question:**

   Consider $L^1[0,1]$. Let X be the set of all polynomials in $L^1[0,1]$, with the usual norm. Give an example if a weak$^*-$ bounded set in X^* that is not bounded. Give complete details of your answer.

   Solution: Consider the linear functionals $T_n : X \mapsto \mathbb{C}$ defined by $T_n(p) = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$. The set $\{T_n : n \geq 1\}$ is weak* bounded but not norm bounded. $\square$

7. **Question:** Let X be a topological vector space with a, balanced and bounded neighborhood of 0. Show that X is metrizable (you may assume that metrization theorem).

   Solution: Let V be a balanced nbhd of 0. Then $\{\frac{1}{n}V : n \in \mathbb{N}\}$ forms a countable local base for X. Hence X is metrizable by metrization theorem. $\square$

8. **Question:**

   On the space $C_b(\mathbb{R})$ of bounded continuous functions, consider the sequence of semi -norms, $p_n(f) = \sup\limits_{[-n,n]} \mid f \mid$. Describe convergent sequences and bounded sets in the topology generated by this family $\{p_n\}_{n\leq 1}$.

   Solution:

$$\begin{aligned} f_k \to f &\Leftrightarrow \forall n \in \mathbb{N}\ p_n(f_k - f) \to 0\ as\ k \longrightarrow \infty \\ &\Leftrightarrow \sup_{x\in[n,-n]} |f_k(x) - f(x)| \to 0 \\ &\Leftrightarrow f_k \to f\ uniformly\ on\ [n,-n],\ for\ every\ n \\ &\Leftrightarrow f_k \to f\ uniformly\ on\ every\ compact\ set\ K \end{aligned}$$

   $f_k \to f$ in $C_b(\mathbb{R}) \Leftrightarrow f_K \to f$ uniformly on every compact subset K of $\mathbb{R}$. Similarly a set is bounded in $C_b(\mathbb{R})$ iff it is uniformly bounded on every compact susbet of $\mathbb{R}$.

   $\square$